Algebraic Topology

I. Introduction

A What is Algebraic Topology

Very generally topology is the study of spaces on which you can discuss

1) continuity of functions and
2) convergence of sequences

we give general definitions later in the course but two main objects of study in topology are manifolds and CW-complexes

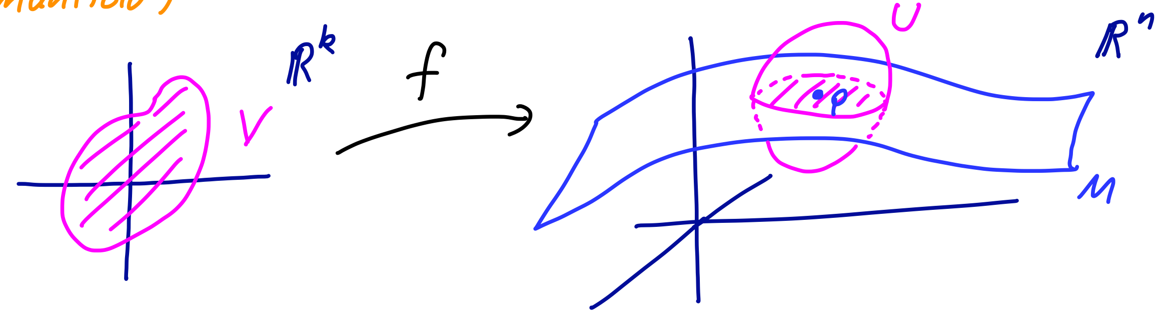
these show up all over math and science as configuration spaces, models of the universe, solution spaces to equations, ...

we focus on manifolds for now (requires less background)

a k-manifold (or manifold of dimension k) M is a subset of $\mathbb{R}^n$ such that for each point $p \in M$ there is

1) an open set U in $\mathbb{R}^n$ containing p
2) an open set V in $\mathbb{R}^k$, and
3) a continuous function $f: V \to U$ such that
   a) f is injective
   b) $\operatorname{im} f = M \cap U$
   c) $(f|_{\operatorname{im} f})^{-1}: M \cap U \to V$ is continuous

(if f is differentiable and $\text{rank}(Df_x) = k$, $\forall x \in V$ then M is a <u>smooth</u> manifold)

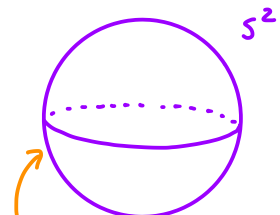


we call f a <u>coordinate chart</u> or <u>local parameterization</u>

Intuitively M is "locally Euclidean", that is if <u>you</u> "lived" in M then you would probably think you were in $\mathbb{R}^k$

(if you were really small compaired to M or had really bad eye sight eg. surface of Earth <u>not</u> $\mathbb{R}^2$)

<u>examples:</u>



surface of the Earth! If you lived here you might think you were in $\mathbb{R}^2$.

1) $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$

is a 2-manifold

(2-manifolds are also called <u>surfaces</u>)

local parameterizations are of the form

$$(x,y) \longmapsto (x, y, \sqrt{1-x^2-y^2})$$

for $(x,y) \in \{x^2+y^2 \leq 1\} \subseteq \mathbb{R}^2$ (need 5 more such charts, what are they?)

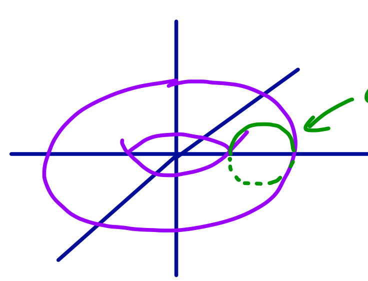
2) $S^k = \{(x_0, \dots, x_k) \in \mathbb{R}^{k+1} : \sum_{i=0}^{k} x_i^2 = 1\}$

is a k-manifold.

3) Torus: image of the map

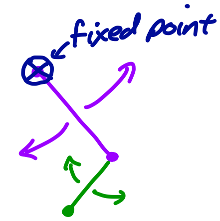
$$f(x,y) = ((3+\cos x)\cos y, (3+\cos x)\sin y, \sin x)$$

is a surface

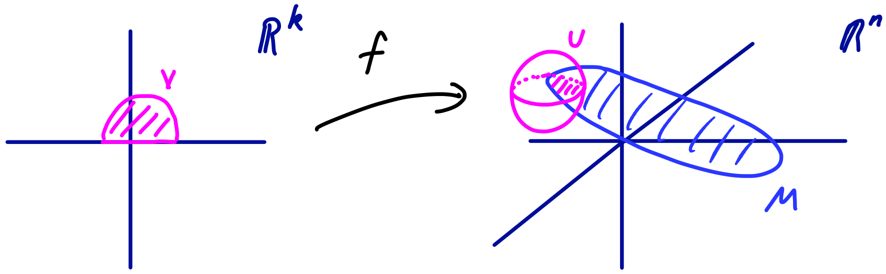


circle in xz-plane rotate about z-axis

note: T^2 is a configuration space
consider the "double pendulum"



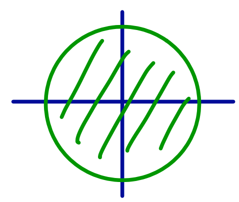
M is called a <u>k-manifold with boundary</u> if we have f, U, V as above except V is an open set in $\mathbb{R}^k_{\geq 0} = \{(x_1, \dots x_k) : x_k \geq 0\}$



examples:

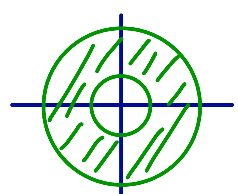
1) $D^k = \{(x_1, \dots, x_k) \in \mathbb{R}^k : x_1^2 + \dots + x_k^2 \leq 1\}$

   k-disk



2) annulus

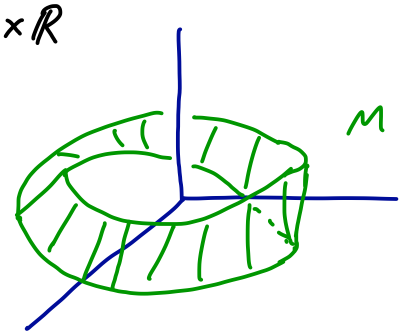
   $A = \{(x,y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 2\}$



3) Möbius band

   image of $f(x,y) = (\underbrace{2 + x\cos y}_{r}, \underbrace{2y}_{\theta}, \underbrace{x \sin y}_{z})$ cylindrical coordinates

   for $(x,y) \in [-1,1] \times \mathbb{R}$



Two manifolds M and N are called <u>homeomorphic</u> if there is a continuous bijection $f: M \to N$ such that $f^{-1}: N \to M$ is also continuous if two manifolds are homeomorphic then we think of them as being the same.

example:

$S^2 \subset \mathbb{R}^3$ the unit sphere and

$S_r^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = r^2\}$ $\quad r>0$

are homeomorphic (what is the map?)

from the topological point of view they are the same (of course "geometrically" they are different, e.g. area)

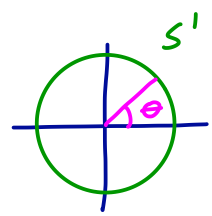
An embedding of one manifold into another is a continuous injective function $f: M \to N$ that is a homeomorphism onto its image.

examples:

1) S^1 = unit circle in $\mathbb{R}^2$

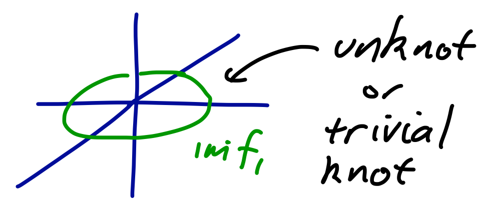
inclusion $i: S^1 \to \mathbb{R}^2$

is an embedding



2) $f_1: S^1 \to \mathbb{R}^3 : \theta \mapsto (\cos\theta, \sin\theta, 0)$

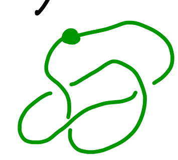
an embedding of S^1 into $\mathbb{R}^3$ is called a knot

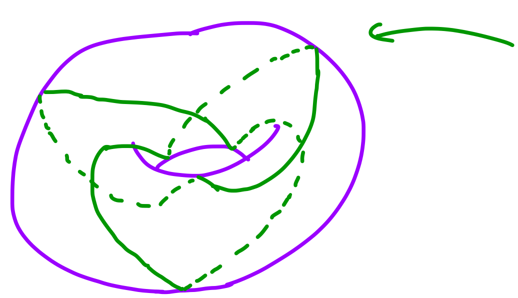


think of it as a piece of string with ends glued together

3) $f_2: S^1 \to \mathbb{R}^3$:

$\theta \mapsto (\cos 3\theta(3+\cos 2\theta), \sin 3\theta(3+\cos 2\theta), \sin 2\theta)$





Main Problems: (same as in other areas of math)

1) list or show how to build all manifolds
2) find ways to distinguish manifolds

} classify

3) study maps between manifolds

   usually restrict to special maps

   examples: • homeomorphisms and
   • embeddings

   (again we want to construct them and distinguish them)

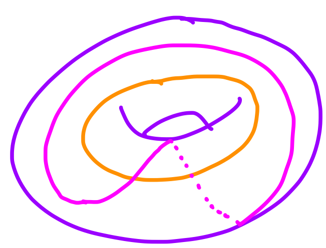
(4) study "structures" on manifolds

   e.g. Riemannian geometry, complex geometry, contact/symplectic geometry, ...) ← different course

There are many surprising relations between all these problems

examples:

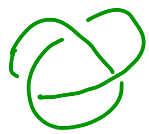
1) use embeddings of curves in surfaces

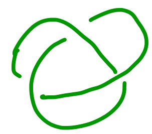


   to understand homeomorphisms of surfaces and to distinguish and build surfaces

2) embeddings of S^1 in S^3 (or $\mathbb{R}^3$) can be used to construct 3 and 4-manifolds

the study of such embeddings is called <u>knot theory</u> and is very interesting on its own

e.g. are  and  the "same"?

what about  and  ?

We will study these problems using algebraic techniques

i.e. Algebraic topology (in a <u>very</u> general sense)

The idea is to build a function

$$\left\{\begin{matrix}\text{something you}\\ \text{want to study}\end{matrix}\right\} \Longrightarrow \left\{\begin{matrix}\text{something algebraic that}\\ \text{is hopefully easier to study}\end{matrix}\right\}$$

e.g. $\{\text{all manifolds}\}$ or $\left\{\begin{matrix}\text{all embeddings}\\ S^1 \hookrightarrow \mathbb{R}^3\end{matrix}\right\}$ or ...

$\mathbb{Z}$ or
set of groups or
set of vector spaces or
set of polynomials or ...

being a function, if two manifolds/embeddings... are sent to to different algebraic things then they are different!

we call such a function an <u>algebraic invariant</u>

It would be even better if the invariant "reflected" properties of the topological objects

some examples we will study

$$\left\{\begin{matrix}\text{topological}\\ \text{spaces}\end{matrix}\right\} \xrightarrow[\text{group}]{\text{fundamental}} \{\text{groups}\}$$

$$X \longmapsto \pi_1(X)$$

we will see: 1) very good invariant of surfaces and knots

2) studying homeomorphisms of surfaces is essentially the same as studying isomorphisms of the fundamental group

(there are some partial generalizations of this to higher dimensions)

3) can use topology to learn things about groups! (this is called "geometric group theory")

The main parts of this course will be

I. Intro. to general topology

including the classification of surfaces using "surgery theory"

II. Brief intro. to groups and group presentations

III. Fundamental group and homotopy theory

IV Covering spaces

but before we really get started, let's see specific example to illustrate the above themes

we will do this through knot theory, much of the first part of this is <u>very</u> "simple" and could be told to highschool students, but later we will see deap connections to algebraic topology!

B. Knot Theory

Recall a knot is the image of an embedding

$$f : S^1 \hookrightarrow \mathbb{R}^3$$

(for now f a smooth embedding)

so $K = \operatorname{im}(f)$ a knot

we say 2 knots K_0 and K_1 are isotopic if there is a smooth map

$$H : S^1 \times [0,1] \longrightarrow \mathbb{R}^3$$

such that

1) $\operatorname{im}(H|_{S^1 \times \{i\}}) = K_i \qquad i = 0, 1$

2) $H|_{S^1 \times \{t\}} : S^1 \to \mathbb{R}^3$ is an embedding $\forall t \in [0,1]$

the idea is that you can smoothly deform K_0 into K_1
(i.e. if K_0 is made out of string, you can move it around to get K_1)

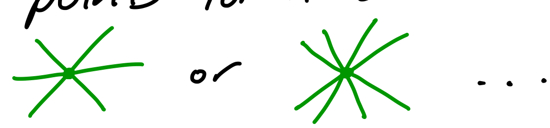
when we say 2 knots are "the same" we mean they are isotopic

Knots are frequently studied via their diagrams

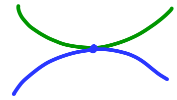
let $p : \mathbb{R}^3 \to \mathbb{R}^2 : (x, y, z) \mapsto (x, y)$ be projection

given a knot K one can show it can be isotoped (by a very small amount) such that

1) $p|_K$ is an immersion (that is derivative non zero)
so you can't see 

2) $p|_K$ has no n-tuple points for $n \geq 3$
don't see 

2) at each douple point the two arcs of K intersect transversely $\longrightarrow$ (tangent vectors of arcs a double point span $\mathbb{R}^2$)

don't see 

(to prove this need "jet transversality" or PL-topology, beyond this course, but hopefully believable)

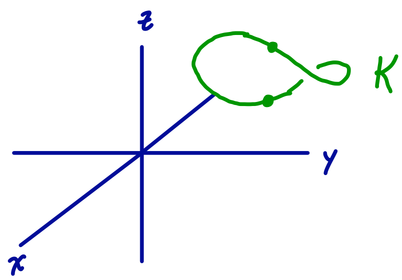
a diagram $D(K)$ of K is

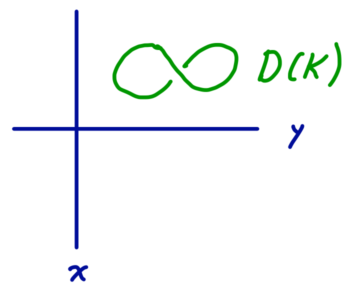
1) $p(K) \subset \mathbb{R}^2$ and

2) at each double point lable which strand goes over the other one
(i.e. which has the greater z-coordinate)

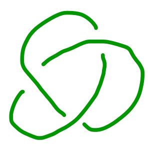
examples:

1)





2)



trefoil

exercise: Show a knot diagram D determines a unique knot in $\mathbb{R}^3$ upto isotopy

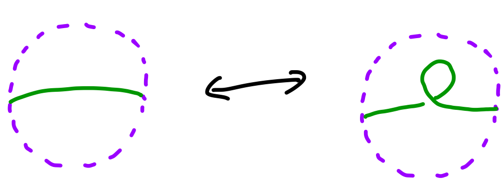
we have an amazing theorem

Reidemeister's Th$^{\underline{m}}$:

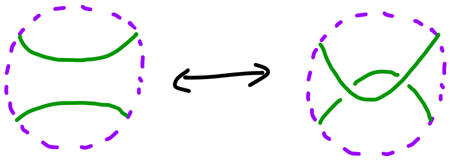
let K_0 and K_1 be knots with diagrams D_0 and D_1

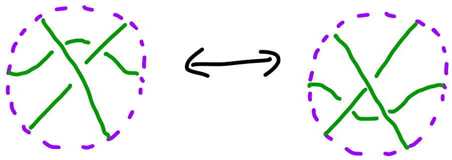
Then K_0 is isotopic to $K_1 \Leftrightarrow D_0$ is related to D_1 by a sequence of

0) deformations where crossings don't change

I) 

(this means, if you see a piece of the diagram looking like one side you can replace it with the other)

II) 

III) 

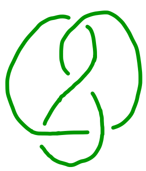
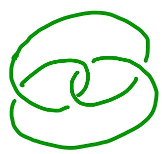
(and reflections of these about coordinate planes)

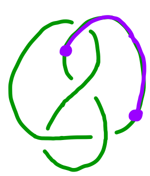
note that ($\Leftarrow$) should be clear

($\Rightarrow$) takes some work

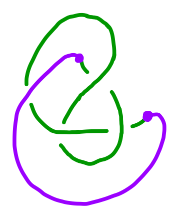
(to prove need "parametric jet transversality" or PL-topology)

example:

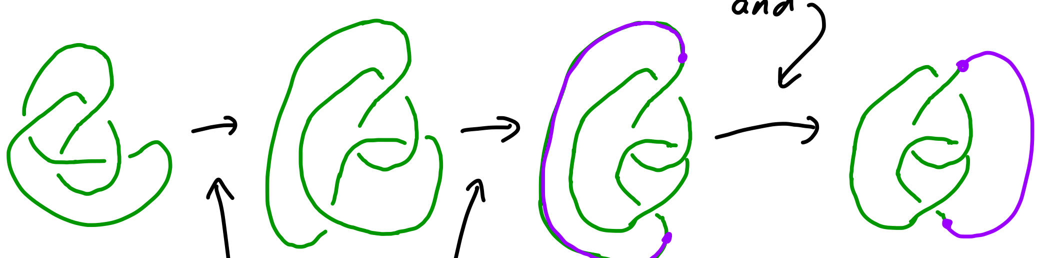
 and  are isotopic

to see this note 

find the Reidemeister moves doing this!



and



just push arcs around not changing crossings

A link is just a disjoint union of knots

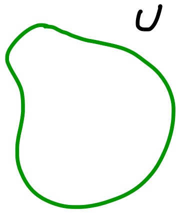
C. Knot coloring

So how can you tell if two knots are different?
here is a very simple way
we say a knot diagram D is 3-colorable if you can color the strands of D with 3 colors so that

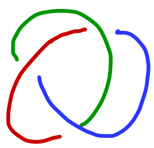
Ⓐ at each crossing either all 3 colors are used or only 1 is used

Ⓑ at least 2 colors are used

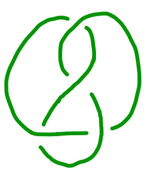
examples:

1)  U

unknot U is not 3-colorable

2)  T

the trefoil T is 3-colorable

exercise:  F

show the "figure 8" knot F is not 3-colorable

Thm 1:

> If one diagram for a knot is 3-colorable then all diagrams are
> (so being 3-colorable is a property of the knot, not just the diagram)

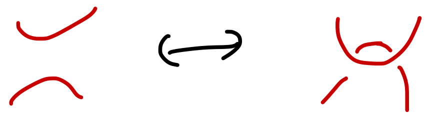
Remark: So from above we see the trefoil T is different from the unknot U and figure 8, F

Proof:
we just need to check 3-colorability is unchanged under Reidemeister moves

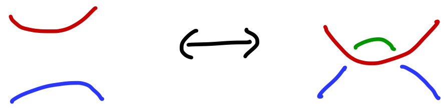
I) 

one color — must only be one color here

so ⓐ true and ⓑ true for one $\Leftrightarrow$ true for the other

II) either 

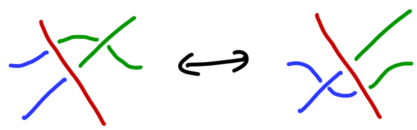
one color — one color

or 

more than one color — more than one color

so ⓐ true and ⓑ true for one $\Leftrightarrow$ true for other

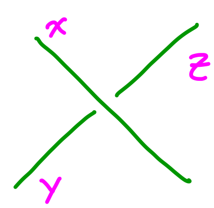
III) lots of cases

here is one 

exercise: check all other cases

more generally we say a diagram (and knot) is <u>p-labelable</u> for p a prime, if we can label the strands with numbers $0, 1, \ldots, p-1$ so that

ⓐ at each crossing, the overcrossing label is the mod p average of the labels of the undercrossings



$2x \equiv y + z \mod p$

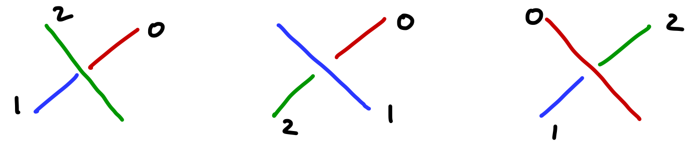
ⓑ at least 2 labels are used

exercise: Prove the analog of Th$^{\underline{m}}$ 1 for p-labeling

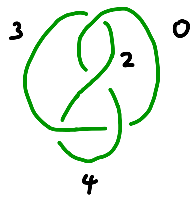
examples:

1) a 3-coloring is a 3-labeling

let red $= 0$ blue $= 1$ green $= 2$



$2 \cdot 2 = 4 \equiv 0 + 1 \mod 3$ $\qquad$ $2 \cdot 1 = 2 \equiv 2 + 0 \mod 3$ $\qquad$ $2 \cdot 0 = 0 \equiv 1 + 2 \mod 3$

2) 

is a 5-labeling of F

so F is not isotopic to U

later in the course we will see how coloring / labeling is related to really cool topology!

(dihedral representations of the fundamental group of the knot complement)